Nonzero-sum expected average discrete-time stochastic games

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A joint work with Qingda Wei

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Contents

Discrete-time Stochastic Game

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- players: $N-$person
- state space: X
- action space: A_m $(1 \le m \le N)$, $A = A_1 \times A_2 \times \cdots \times A_N$
- history: $H_0 := X$, $H_n = (X \times A)^n \times X$ $(n \ge 1)$
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If $N = 2$ and $r_1 = -r_2$, zero-sum.

Strategy

- Randomized history-dependent strategy: $\pi_n^m(\cdot|h_n)$ $(1 \leq m \leq N, n \geq 0)$ stochastic kernels on A_m
- Stationary Markov strategy: if there exists a stochastic kernel $\phi^m \in \Phi_m$ such that $\pi_n^m(\cdot|h_n) = \phi^m(\cdot|x_n)$ for all $h_n \in H_n$ and $n \geq 0$.

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- Stationary almost Markov strategy: if there exists a stochastic kernel $\widetilde{\phi}^m \in \widetilde{\Phi}_m$ such that $\pi_n^m(\cdot|h_n) = \widetilde{\phi}^m(\cdot|x_{n-1}, x_n)$ for all $h_n \in H_n$ and $n \geq 0$, where $x_{-1} := x_0$.

Optimality Criteria

- discounted payoff $J_m(x,\pi) := E_x^\pi \left[\sum_{t=0}^{\infty} \alpha^t r_m(x_t, a_t)\right]$ (discount factor $\alpha < 1$)
- finite horizon (for any fixed T) payoff $J_m(x,\pi) := E_x^\pi \left[\sum_{t=0}^{T} r_m(x_t, a_t)\right]$
- expected average payoff $J_m(x,\pi) := \liminf_{n\to\infty} \frac{1}{n} E_x^\pi \left[\sum_{t=0}^{n-1} r_m(x_t, a_t)\right]$

Nash Equilibrium

For any strategy profile $\pi = (\pi^1, \ldots, \pi^N) \in \Pi$ and any strategy $\overline{\pi}^m \in \Pi_m$ ($m \in \mathbb{I}$), we use the notation $[\pi_{-m}, \overline{\pi}^m]$ to denote π with π^m replaced by $\overline{\pi}^m$.

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Definition 1.1

A strategy profile $\pi^* = (\pi^{*1}, \dots, \pi^{*N}) \in \Pi$ is said to be a Nash equilibrium for the game model $\mathcal{G}$ under the expected average payoff criterion if

$$J_m(x, \pi^*) \geq J_m(x, [\pi^*_{-m}, \pi^m])$$

for all $x \in X$, $\pi^m \in \Pi_m$ and $1 \leq m \leq N$.

History

- Nash (1950 PNAS)
- Shapley (1953 PNAS)
- Fink (1964) and Takahashi (1964)
- ...
- Survey papers: Dutta and Sundaram (1998) Jaśkiewicz and Nowak (2018)

Model

- players: $N-$person
- state space: X is a Borel space
- action space: A_m $(1 \leq m \leq N)$, $A = A_1 \times A_2 \times \cdots \times A_N$ are Borel space
- history: $H_0 := X$, $H_n = (X \times A)^n \times X$ $(n \geq 1)$
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Game model $\mathcal{G} =: \left\{X, (A_m, \{A_m(x) \subseteq A_m, x \in X\}, r_m(x, a))_{m \in \mathbb{I}}, Q(\cdot|x, a)\right\}$.

Expected average payoff $J_m(x, \pi) := \liminf_{n \to \infty} \frac{1}{n} E_x^{\pi} \left[\sum_{t=0}^{n-1} r_m(x_t, a_t)\right]$

Assumptions

Assumption

- *(i) There exist a measurable function* $w \geq 1$ *on* X*, a Borel set* $C \subset X$*, and constants* $\rho \in (0, 1)$*,* $b > 0$ *such that*

$$\int_X w(y) Q(dy|x, a) \leq \rho w(x) + b I_C(x) \text{ for all } (x, a) \in X \times A \text{ and } \sup_{x \in C} w(x) < \infty,$$

where $I_C(\cdot)$ *denotes the indicator function of the set* C*.*

- *(ii) There exist a constant* $\xi \in (0, 1)$ *and a probability measure* γ *concentrated on the set* C *such that* $Q(D|x, a) \geq \xi \gamma(D)$ *for any Borel set* $D \subset C$*,* $x \in C$ *and* $a \in A(x)$*.*

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- *(iii) There exists a constant* $M > 0$ *such that* $|r_m(x,a)| \leq Mw(x)$ $1 \leq m \leq N$*.*
- *(iv) For each* $x \in X$ *and* $1 \leq m \leq N$*, the set* $A_m(x)$ *is compact.*
- *(v) For each* $x \in X$ *and* $1 \leq m \leq N$*,* $r_m(x,a)$ *is continuous in* $a \in A(x)$*.*

Assumptions (continued)

Assumption

(vi) There exist functions $g_1, \dots, g_p \in Car(X \times A)$ satisfying $g_k \in [0,1]$, $\sum_{k=1}^{p} g_k(x,a) = 1$ for all $(x,a) \in K$ and stochastic kernels $\mu_1, \dots, \mu_p \in \mathbb{P}(\mathcal{B}(X)|X)$ such that

$$Q(\cdot|x,a) = \sum_{k=1}^{p} g_k(x,a)\mu_k(\cdot|x) \text{ and } \int_X w(y)\mu_k(dy|x) < \infty$$

for $k = 1, \dots, p$. Moreover, there exists a probability measure μ^* on $\mathcal{B}(X)$ such that each $\mu_k(\cdot|x)$ $(1 \le k \le p)$ is absolutely continuous with respect to μ^* for all $x \in X$.

Main Result

Main Theorem

Under the above Assumptions, there exists a stationary almost Markov Nash equilibrium for the game model $\mathcal{G}$.

Remark

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- *(i) In a number of applications of discrete-time stochastic games in economics, the state space is a subset of Euclidean space, such as an interval.*
- *(ii) Levy (2013 Econometrica) gave a counterexample of a discounted game with uncountable state space, finite actions that stationary Markov perfect equilibria do not exist.*

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- *(ii) Levy (2013 Econometrica) gave a counterexample of a discounted game with uncountable state space, finite actions that stationary Markov perfect equilibria do not exist.*
- *(iii) Küenle(1999), Benitez-Medina (2001), Nowark and Altman (2002), Jaśkiewicz and Nowak (2015) etc. additive reward and additive transitions*

$$c_i(x, a_1, a_2) = c_{i1}(x, a_1) + c_{i2}(x, a_2), \quad q(x, a_1, a_2) = q_{i1}(x, a_1) + q_2(x, a_2)$$

Assumption (vi') There exist functions $g_1, \ldots, g_p \in Car(X \times A)$ *satisfying* $g_k \in [0, 1]$, $\sum_{k=1}^{p} g_k(x, a) = 1$ *for all* $(x, a) \in K$ *and stochastic kernels* $\mu_1, \ldots, \mu_p \in \mathbb{P}(\mathcal{B}(X))$ *such that*

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for $k = 1, \ldots, p$. *Moreover, there exists a probability measure* μ^* *on* $\mathcal{B}(X)$ *such that each* $\mu_k(\cdot)$ $(1 \le k \le p)$ *is absolutely continuous with respect to* μ^* *for all* $x \in X$.

Priliminary

Lemma (Average-Cost Optimality Equation)

$$\rho^* + h(x) = \inf_{A(x)} \left[r(x,a) + \int h(y) Q(dy|x,a) \right]$$

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Note: $\rho^* := \lim_{n\to\infty} \frac{1}{n} E_x^\pi \left[\sum_{t=0}^{n-1} r(x_t,a_t)\right]$, $h(x) := E_x^\pi \left[\sum_{t=0}^{\infty} (r(x_t,a_t) - \rho^*)\right]$

$$h(x) = r(x,a) - \rho^* + E_x^\pi \left[\sum_{t=1}^{\infty} (r(x_t,a_t) - \rho^*) \right] = r(x,a) - \rho^* + \int h(y) Q(dy|x,a).$$

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Poisson Equation

Priliminary

Lemma (Discounted-Cost Optimality Equation)

$$V_\alpha^*(x) = \inf_{A(x)} \left[r(x,a) + \alpha \int V_\alpha^*(y) Q(dy|x,a) \right]$$

$$\begin{aligned} V_\alpha(x) =& E_x^\pi \left[\sum_{t=0}^{\infty} \alpha^t r(x_t, a_t) \right] \\ =& r(x,a) + \alpha E_x^\pi \left[\sum_{t=1}^{\infty} \alpha^{t-1} r(x_t, a_t) \right] \\ =& r(x,a) + \alpha \int V_\alpha(y) Q(dy|x,a) \end{aligned}$$

Priliminary

Lemma (Abelian Theorem)

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}E_x^{\pi}\left[\sum_{t=0}^{n-1}r(x_t,a_t)\right] &\le \liminf_{\alpha\to 1}(1-\alpha)E_x^{\pi}\left[\sum_{t=0}^{\infty}\alpha^t r(x_t,a_t)\right]\\ &\le \limsup_{\alpha\to 1}(1-\alpha)E_x^{\pi}\left[\sum_{t=0}^{\infty}\alpha^t r(x_t,a_t)\right]\\ &\le \limsup_{n\to\infty}\frac{1}{n}E_x^{\pi}\left[\sum_{t=0}^{n-1}r(x_t,a_t)\right]\end{aligned}$$

Priliminary

Define relative value function $h_\alpha(x) := V_\alpha^*(x) - V_\alpha^*(z)$ (with $z \in X$ be a fixed point), and $\rho_\alpha^* := (1-\alpha)V_\alpha^*(z)$.

$$\rho_\alpha^* + h_\alpha(x) = \inf_{A(x)} \left[r(x,a) + \int h_\alpha(y) Q(dy|x,a) \right].$$

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Let $\alpha \to 1$,

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Uniform estimation of $h_\alpha(x)$

Priliminary

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Extended game model

$$\widetilde{\mathcal{G}} := \left\{\widetilde{X}, \left(\widetilde{A}_m, \{\widetilde{A}_m(\widetilde{x}) \subset \widetilde{A}_m, \widetilde{x} \in \widetilde{X}\}, \widetilde{r}_m(\widetilde{x}, a)\right)_{m \in \mathbb{I}}, \widetilde{Q}(\cdot|\widetilde{x}, a)\right\}$$

with the state space given by $\widetilde{X} := X \times X$, the action space $\widetilde{A}_m := A_m$, the reward function $\widetilde{r}_m(x^-, x, a) := r_m(x,a)$ for all $(x^-, x) \in \widetilde{X}$, and the transition law given by $\widetilde{Q}(D_1 \times D_2 | x^-, x, a) := \delta_x(D_1) Q(D_2|x,a)$ for all $(x^-, x) \in \widetilde{X}$, $D_1, D_2 \in \mathcal{B}(X)$.

Main Result

Main Theorem

Under Assumptions (i)-(vi), there exists a stationary almost Markov Nash equilibrium for the game model $\mathcal{G}$.

Idea of proof

Step 1: Construct six static games.

(1) a game model corresponding to the discounted model;
(2) a game model corresponding to the discounted model subtracting the value at a fixed point (i.e. relative discounted model);
(3) a game model coming from passing to the limit as the discount factor goes to one;

(4) (5) (6) are versions of (1) (2) (3) in the extended state space.

Idea of proof

Step 2: Uniform estimate

Assumption

- *(i) There exist a measurable function* $w \geq 1$ *on* X*, a Borel set* $C \subset X$*, and constants* $\rho \in (0,1)$*,* $b > 0$ *such that*

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  where $I_C(\cdot)$ *denotes the indicator function of the set* C*.*
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Idea of proof

Step 2: Uniform estimate

Lemma (Uniform ω-geometric ergodicity)

Suppose that Assumptions (i) (ii) holds. Then for any $\varphi \in \Phi$, there exist a probability measure ν_φ on X, constants $R > 0$ and $\eta \in (0,1)$ (independent of φ) such that

$$\left| E_x^\varphi[u(x_n)] - \int_X u(y)\nu_\varphi(dy) \right| \leq \|u\|_w R\eta^n w(x)$$

for all $u \in B_w(X)$, $x \in X$ and $n \geq 0$.

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for all $u \in B_w(X)$, $x \in X$ and $n \geq 0$.

Theorem

Suppose that Assumptions (i) (ii) is satisfied. Then for each $\widetilde{\phi} \in \widetilde{\Phi}$, there exist a probability measure $\widetilde{\nu}_{\widetilde{\phi}}$ on $\widetilde{X}$, constants $\widehat{R} > 0$ and $\widehat{\eta} \in (0,1)$ (independent of $\widetilde{\phi}$) such that

$$\left| \widetilde{E}_{\widetilde{x}}^{\widetilde{\phi}}[\widetilde{u}(\widetilde{x}_n)] - \int_{\widetilde{X}} \widetilde{u}(\widetilde{y})\widetilde{\nu}_{\widetilde{\phi}}(d\widetilde{y}) \right| \leq \|\widetilde{u}\|_{\widetilde{w}} \widehat{R}\widehat{\eta}^n \widetilde{w}(\widetilde{x})$$

for all $\widetilde{u} \in B_{\widetilde{w}}(\widetilde{X})$, $\widetilde{x} \in \widetilde{X}$ and $n \geq 0$.

Idea of proof

Step 3: Applying Caratheodory's Convexity Theorem and Filippov's Measurable Implicit Function Theorem.

Thank you !

Lemma (Caratheodory's Convexity Theorem)

In an n-dimensional vector space, every vector in the convex hull of a nonempty set can be written as a convex combination using no more than $n + 1$ *vectors from the set.*

Priliminary

Lemma (Filippov's Measurable Implicit Function Theorem)

Assume F is a lower measurable set-valued map from S into nonempty compact subsets of X, Y is a metric space, $u : S \times X \to Y$ is a function such that, $u(s, \cdot)$ is continuous on X and for every $s \in S$, $u(\cdot, x)$ is measurable for every $x \in X$. Suppose that there is a measurable function $g : S \to Y$ such that $g(s) \in \{u(s, x) : x \in F(s)\}$ for every $s \in S$. Then there exists a measurable selector of F such that

$$g(s) = u(s, f(s)) \quad \textit{for every } s \in S.$$

Priliminary

Lemma (Application of Filippov's Measurable Implicit Function Theorem)

Let μ be a nonatomic Borel probability measure on X. Assume that $q_j (j = 1, 2, \dots, l)$ are Borel measurable transition probabilities from X to X and for every j and $x \in X$, $q_j(\cdot|x) \ll \mu$. Let $w^0 : X \to R^n$ be a Borel measurable mapping such that $w^0 \in co\mathcal{NP}(x)$ for each $x \in X$. Then there exists a Borel measurable mapping $v^0 : X \times X \to R^n$ such that $v^0(x, y) \in \mathcal{NP}(x)$ for all $x, y \in X$ and

$$\int_X w^0(y)q_j(dy|x) = \int_X v^0(x, y)q_j(dy|x), \quad j = 1, 2, \dots, l.$$

Moreover, there exists a Borel measurable mapping $\phi : X \times X \to Pr(A)$ such that $\phi(x, y) \in \mathcal{P}(x)$ for all $x, y \in X$.

Lemma (Fan-Kakutani-Glicksberg Fixed Point Theorem)

Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let the correspondence $\phi : K \twoheadrightarrow K$ have closed graph and nonempty convex values. Then the set of fixed points of ϕ is compact and nonempty.